Elements of Reinforcement Learning

- Policy: way learning algorithm behaves (mapping from state to action)
- Reward function: Mapping of state action pair to reward or cost
- Value function: long term reward, total weighted or unweighted reward in present and future
- Model: mimic behavior of environment
Consider n-armed bandit problem: at every instant must choose one of n actions with the goal of maximizing rewards.

Expected reward for action \( a \), \( Q^*(a) \) and estimated value of \( \tau \)th play \( Q_\tau(a) \). Set

\[
    Q_\tau(a) = \sum r_i / k_a
\]

where \( k_a \) is number of times action \( a \) taken

Exploration versus exploitation

How to choose action \( a \)
Policies for n-armed bandit problem

- **Greedy policy**: $Q_t(a^*) = \max_a Q_t(a)$. No exploration, initially does well, but poor long term performance.
- **$\epsilon$-greedy policy**: same as greedy policy, but with prob. $\epsilon$ randomly choose policy. Some exploration, does better than greedy policy.
- **Softmax Action selection**: weight actions probabilistically with temperature parameter (Gibbs distribution).
- **Reinforcement methods**: keep track of payoffs (reinforcement as opposed to action-value method).
- **Pursuit methods**: use both action-value estimates and action preferences.
Summary of example

- **Exploitation versus Exploration**
  - Algorithms presented
  - What is the proper balance?

- **Learning schemes**
  - Supervised learning: instructed what to do
  - Evaluative learning: try different actions and observe rewards (allows more control of environment)

- **Non associative learning**: trial and error learning not associated with situation or state of the problem (only one state)
Reinforcement Learning Model

- Exploration versus exploitation
- Learning can be slow
Finite Markov Decision Processes

- **Parameters**
  - **State:** $X(n) = x(n)$, $N$ states
  - **Action:** $A(n) = a_{ik}$ (action from state $i$ performing action $k$)
  - **Transition probability:** $p_{ij}(a) = P(X(n+1) = j \mid X(n) = i, A(n) = a)$
  - **Cost:** $r(i,a,j)$ and discount factor $\gamma$, with $0 \leq \gamma < 1$
  - **Policy:** $\pi = \{u(0), u(1), \ldots\}$, policy mapping states into actions (stationary and nonstationary)
Value Functions

- Cost or value function (infinite horizon, discounted)
  \[ J^\pi(i) = \mathbb{E} \left( \sum \gamma^n r(x(n), u(x(n)), x(n+1)) \middle| x(0)=i \right) \]
  averaged over Markov chain \( x(1), x(2), \ldots \)

- Action-value function
  \[ Q^\pi(i, a) = \mathbb{E} \left( \sum \gamma^n r(x(n), u(x(n)), x(n+1)) \middle| x(0)=i, a(0)=a \right) \]
  averaged over Markov chain \( x(1), x(2), \ldots \)

Find policy \( \pi \) that minimizes \( J^\pi(i) \) for all initial states \( i \)
Recursive expression for value function

\[ J_\pi(i) = E \left( \sum \gamma^n r(x(n), u(x(n)), x(n+1)) | x(0) = i \right) \]
\[ = E(r(i, u(i), j) + \gamma \sum \gamma^n r(x(n+1), u(x(n+1)), x(n+2)) | x(1) = j) \]
\[ = \sum_a \sum_j \pi(i, a) p_{ij}(a) (r(i, a, j) + \gamma J_\pi(j) ) \]

Bellman equation for \( J_\pi \) allows for calculation of value function for policy \( \pi \).
Equation can be solved iteratively or directly.
Optimal Value Function

Want to find optimal policy to maximize value function

\[ J^*(i) = \max_{\pi} J^\pi(i) \]

Can express optimal value function in terms of action-value function as

\[ J^*(i) = \max_{u(i)} Q^*(i,u(i)) \]

where \( Q^*(i,u(i)) = \max_{\pi} Q^\pi(i,u(i)) \)

Then can find a recursive expression for \( J^*(i) \) by expanding RHS of equation similar to method found in previous slide for value function.
Use dynamic programming, can formulate cost function in terms of Bellman’s Optimality equation

\[ J^*(i) = \max_u \ E_{x(i)} [r(i,u(i),x(i)) + \gamma J^*(x(i))] \]

Current cost: \( c(i,u(j)) = E_{x(i)} [r(i,u(i),j)] = \sum_{j=1,N} p_{ij} r(i,u(i),j) \)

Rewrite Bellman’s equation

\[ J^*(i) = \max_u [c(i,u(i)) + \gamma \sum_{j=1,N} p_{ij} J^*(j)] \]

System of \( N \) equations with (equation/ state) and minimization
Policy Evaluation and Improvement

- Policy Evaluation: For a given policy we can iteratively compute value function
  \[ J_{k+1}^{\pi}(i) = \sum_a \sum_j \pi(i,a) p_{ij}(a) (r(i,a,j) + \gamma J_k^{\pi}(j)) \]
  Iterative algorithm converges.
- Policy Improvement: Q function can be expressed iteratively as
  \[ Q^{\pi}(i,a) = c(i,a) + \gamma \sum_{j=1,N} p_{ij}(a) J^{\pi}(j) \]
  u is said to be greedy with respect to \( J^u(i) \) if
  \[ u(i) = \max_a Q^{\pi}(i,a) \text{ for all } i \]
Policy Iteration

1) Policy evaluation: $J^u (i)$
Cost to go function needs recomputation

$$J^u_n(i) = c(i,u_n(i)) + \gamma \sum_{j=1,N} p_{ij} (u_n(i)) J^u_n(j)$$

Solve set of $N$ linear equations directly or iteratively.

2) Policy improvement: $u_{n+1}(i) = \arg\max_a Q^u_n(i,a)$
Value Iteration

- Initialization: start with initial value $J_0(i)$
- Iterate: $Q(i,a) = c(i,a) + \gamma \sum_{j=1,N} p_{ij} J_n(j)$
  \[ J_{n+1}(i) = \max_a Q(i,a) \]
- Continue until $|J_{n+1}(i) - J_n(i)| < \varepsilon$
- Compute policy: $u^* = \arg\max_a Q(i,a)$
Dynamic Programming Comments

- Number of states often grows exponentially as number of state variables. (Bellman’s curse of dimensionality)
- For large state spaces it is infeasible to search entire state space to perform DP steps. Asynchronous DP used where partial searches and updates are made of state space.
- DP programs run polynomially in number of states and actions.
- GPI (Generalized Policy Iteration) often used instead of PI where Policy Evaluation and Policy Improvement done together.
- DP assumes complete knowledge of environment.
Approximate Dynamic Programming

- Incomplete information (do not know Markov transition probabilities)
- Curse of dimensionality
- Opt for suboptimal policy where $J^*(i)$ replaced by approximations of $J^*(i)$ that can consist of table lookup or parameterized by set of weights
- Use Monte Carlo simulations to learn policy
- Q learning
Q Learning Algorithm

- Define Q function
  \[ Q^*(i,a) = \sum_{j=1,N} p_{ij}(a) \left( r(i,a,j) + \gamma \max_b Q^*(j,b) \right) \]
  \[ J^*(i) = \max_a Q^*(i,a) \]

- Use iterative learning to learn Q function
  \[ Q_{n+1}(i,a) = (1 - \mu(i,a)) Q_n(i,a) + \mu(i,a)(r(i,a,j) + \gamma J_n(j)) \]
  where j is random successor state with
  \[ J_n(j) = \max_b Q_n(i,b) \]

- Monte Carlo Simulations: update only applies to current state-action pair all other pairs are not updated
Q Learning Comments

- Convergence Theorem: Q Learning algorithm converges almost surely to optimal Q function given certain conditions on step size (stochastic approximation conditions) and all state pairs are visited infinitely often.
- Representations: Table lookup works well, but networks parameterized by weights often learn very slowly.
- Exploration vs. exploitation: ensure all state-action pairs are explored while also minimizing cost to go function.
Temporal Difference Learning

- Given a learning sequence where a termination occurs and a reward is given how do we learn?
- Credit assignment to each training input in the sequence can be performed using temporal difference learning
- Iterative learning algorithms can then be established with inputs and target outputs
- Class of TD(\(\lambda\)) algorithms where \(0 \leq \lambda \leq 1\)
- Learning much slower than supervised learning
Reinforcement Learning Applications

- Backgammon
- Navigation
- Elevator control
- Helicopter control
- Computer network routing
- Sequential detection
- Dynamic channel allocation (cellular system)
References